ELSEVIER

Journal of Geometry and Physics 32 (1999) 189–210

JOURNAL OF GEOMETRY AND PHYSICS

Twistor intergral representations of fundamental solutions of massless field equations

Kazuhiko Aomoto [a], Yoshinori Machida [b,*]

[a] *Department of Mathematics, School of Science, Nagoya University, Chikusa-ku, Nagoya 464-01, Japan*

[b] *Numazu College of Technology, 3600 Ooka, Numazu-shi, Shizuoka 410, Japan*

Received 24 November 1998

Abstract

We consider the general dimensional (complex) Minkowski spaces and the extended twistor spaces. We show that the fundamental solutions of the complex wave or Laplace equations are explicitly represented by the integrals of some closed forms on the twistor spaces. The closed form is defined from labeled trees explained in graphs theory, and is written, as the cohomology class, by the linear combination of the logrithmic forms on some hyperplane configuration complement in some complex affine space. © 1999 Elsevier Science B.V. All rights reserved.

Subj. Class.: Spinors; Twistors

1991 MSC: 53A30, 53C65, 58A10, 44A12

Keywords: Twistor theory; Fundamental solution; Integral representation; Labeled tree

0. Introduction

Twistor theory constructed by Penrose is to discuss the relation between field equations on the four-dimensional (complex) space–time and geometric objects on the three-dimensional complex manifold called the twistor space. For example, solutions of equations of linear fields, such as the massless Klein–Gordon field with spin 0, the massless Dirac field with spin $\frac{1}{2}$, are represented by integral representations or cohomologies of functions on the twistor space. Solutions of self-dual Yang–Mills equations (i.e., instantons), which are equations of nonlinear gauge fields, are represented by holomorphic vector bundles on the twistor space.

* Corresponding author.

PII: S0393-0440(99)00022-4

There are various generalizations of the twistor theory. In this paper, we consider the general dimensional (complex) Minlowski spaces and the extended twistor spaces. We represent massless fields, i.e., solutions of the conformally invariant Klein–Gordon equations (i.e., the wave equations) and Dirac equations (i.e., the Weyl equations) by integral representations of functions on the twistor spaces. Especially, we show that the (complex) fundamental solutions, i.e., the propagator functions are explicitly represented by the integrals of some closed forms on the twistor spaces (Theorem 1 in Section 3 and Theorem 2 in Section 4).

In the four-dimensional twistor theory by Penrose, there are the following double fibrations. Let $\mathbf{T}$ be the four-dimensional complex vector space, and let $\mathbf{P}(\cong P^3(\mathbf{C}))$ (of dimension 3) be the set of all one-dimensional complex vector subspaces in $\mathbf{T}$, $\mathbf{M}$ $(\cong G_{2,4}(\mathbf{C}))$ (of dimension 4) the set of all two-dimensional complex vector subspaces in $\mathbf{T}$ and $\mathbf{F}$ (of dimension 5) the set of all flags consisting of one-dimensional and two-dimensional complex subspaces in $\mathbf{T}$. Then we have the following double fibrations:

$$\begin{tikzcd} & \mathbf{F} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathbf{P} & & \mathbf{M} \end{tikzcd}$$

where μ, ν are the natural projections and each fiber of $\mathbf{P}$, $\mathbf{M}$ is $P^2(\mathbf{C})$, $P^1(\mathbf{C})$. This diagram is invariant under $SL(4, \mathbf{C})$. On $\mathbf{M}$, the natural (complex) conformal structure is defined in consideration of $SL(4, \mathbf{C})/\mathbf{Z}_2 \cong SO(6, \mathbf{C})$. The space $\mathbf{P}$ is called the twistor space of $\mathbf{M}$.

We consider the following (affine) local coordinate system in the big affine cell $\mathbf{M}^I \subset \mathbf{M}$:

$$z = (z^{ij})_{i,j=1,2} \in \mathbf{C}^{2\times 2} \cong \mathbf{C}^4 \cong \mathbf{M}^I,$$

and the projective coordinate system in the big cell $\mathbf{P}^I \subset \mathbf{P}$:

$$[u^1, u^2, v_1, v_2](v_1, v_2 \neq 0) \in \mathbf{P}^I.$$

Then the corresponding local coordinate system in the big cell $\mathbf{F}^I \subset \mathbf{F}$ is represented by $(z^{ij}, [v_k])$. The link between $\mathbf{M}$ and $\mathbf{P}$ is given by the following twistor equations:

$$u^j = \sum_{k=1}^{2} \mathrm{i} z^{jk} v_k \quad (j = 1, 2).$$

The (flat) complex metric associated with the diagram is represented as

$$\mathrm{d}s^2 = \mathrm{d}z^{11}\,\mathrm{d}z^{22} - \mathrm{d}z^{12}\,\mathrm{d}z^{21}.$$

The complex wave or Laplace equation compatible with this metric is

$$\Box\phi = \frac{\partial^2\phi}{\partial z^{11}\partial z^{22}} - \frac{\partial^2\phi}{\partial z^{12}\partial z^{21}} = 0.$$

In physics, the equation above represents the motion of equation of a free scalar field ϕ with spin 0 and mass 0, i.e., the massless Klein–Gordon equation.

The (complex) fundamental solution Ψ to this equation is

$$\Psi(z) = \frac{1}{z^{11}z^{22} - z^{12}z^{21}}.$$

We take on $\mathbf{P}$ a rational function

$$f([u^i, v_j]) = \frac{1}{u^1 u^2}.$$

Then the fundamental solution Ψ has a twistor integral representation by f:

$$\Psi(z) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma \subset \nu^{-1}(z)} f\left(\left[\sum_{k=1} 2\mathrm{i} z^{jk} v_k, v_i\right]\right)(v_1\,\mathrm{d}v_2 - v_2\,\mathrm{d}v_1),$$

where γ is a 1-cycle around the point $-z^{12}/z^{11}$ in the fiber $\nu^{-1}(z) \cong P^1(\mathbf{C})$ over $z \in \mathbf{M}^I$. The conclusion follows from the residue theorem. In detail, see [14,13]. The purpose of our paper is to extend this fact to general dimensional twistor theory. When we restrict the argument above to $S^4 \subset \mathbf{M}$, Atiyah ([1,10,14]) associated the Green function for the Laplacian at $x \in S^4$ with the Serre class of $\mu\nu^{-1}(x) \subset \mathbf{P}$.

This paper is organized as follows.

In Section 1, we describe the twistor integral representations of the solutions of the complex wave or Laplace equation on $\mathbf{M}^I$.

Both Sections 3 and 4 are the main parts of this paper.

In Section 3, we study the twistor integral representation of the fundamental solution of this equation in the even dimensional cases. We consider labeleled trees explained in graph theory and define the different form on $\mathbf{P}^I$ associated with each labeled tree. Next we define the differential form ω by summing up all the differential forms associated with each labeled tree. Then we show that ω is closed. Integrating ω on an appropriate cycle in the fiber over each point of $\mathbf{M}^I$, we obtain the twistor integral representation of the fundamental solution of this equation.

In Section 4, we treat the odd dimentional cases. We obtain the twistor integral representation of the fundamental solution of this equation by a reduction from the even dimensional cases.

1. Twistor space of n-dimensional space–time

1. Let M^I be the n-dimensional flat space–time, i.e., the n-dimensional Minkowski space $\mathbf{R}^n_1$. Let $\mathbf{M}^I$ be the n-dimensional flat complex space–time which complexifies both M^I and the metric, i.e., the n-dimensional complex Minkowski space $\mathbf{C}^n$.

We suppose until Section 4 that the dimensions of M^I and $\mathbf{M}^I$ are even dimensional, say $n = 2m$.

We take the coordinate system $(z_i, z_{m+i}) = (z_1, \ldots, z_m, z_{m+1}, \ldots, z_{2m})$ on $\mathbf{M}^I$ such that the complex metric $\mathrm{d}s^2$ is represented in the form

$$\mathrm{d}s^2 = 2\left(\sum_{i=1}^{m} \mathrm{d}z_i\, \mathrm{d}z_{m+i}\right).$$

In this paper we discuss our problems in terms of the local or affine chart. This is sufficient to solve physical problems. We do not treat either global or cohomological problems as is discussed in [3].

2. Let $\mathbf{P}^I$ be the set of all maximal isotropic subspaces in $\mathbf{M}^I$ which do not necessarily include the origin, i.e., the set of all totally null affine m-places in $\mathbf{M}^I$. Here by a totally null plane we mean a plane with the property that any two vectors tangent to the plane are orthogonal.

We call $\mathbf{P}^I$ the twistor space of $\mathbf{M}^I$. See [2,12].

The following proposition is easy to see (cf. [5]).

Proposition 1. *A generic element belonging to $\mathbf{P}^I$, i.e., a generic totally null affine m-plane in $\mathbf{M}^I$ is represented by the following twistor equations:*

$$u_i = z_i + \sum_{j=1}^{m} v_{ij} z_{m+j} \quad (i = 1, \ldots, m),$$

where u_i, $v_{ij} \varepsilon \mathbf{C}$, $v_{ij} = -V_{ji}$.

The twistor equations play an important role in linking $\mathbf{M}^I$ to $\mathbf{P}^I$. The equations are characterized by the following holonomic system of linear partial differential equations of first order:

$$\frac{\partial \Phi_i}{\partial z_j} = 0\ (j \neq i), \qquad \frac{\partial \Phi_i}{\partial z_j} = \frac{\partial \Phi_j}{\partial z_j},$$

$$\frac{\partial \Phi_i}{\partial z_{m+j}} + \frac{\partial \Phi_j}{\partial z_{m+i}} = 0\ (i < j), \quad \frac{\partial \Phi_i}{\partial z_{m+i}} = 0\ (i, j = 1, \ldots, m),$$

$$\Longleftrightarrow \Phi_i = u_i - \left(z_i + \sum_{j=1}^{m} v_{ij} z_{m+j}\right) \quad (i, 1, \ldots, m),$$

We remark that we can take (u_i, v_{ij}) as generic parameters of $\mathbf{P}^I$. Therefore, the dimension of $\mathbf{P}^I$ is $m(m+1)/2$.

Let $\mathbf{F}^I$ be the incidence relation between $\mathbf{M}^I$ and $\mathbf{P}^I$, i.e.,

$$\mathbf{F}^I = \{(z, u) \in \mathbf{M}^I \times \mathbf{P}^I | z \in u\}.$$

The dimension of $\mathbf{F}^I$ is $m(m+3)/2$.

3. Let $\mathbf{M}$ be the conformal compactification of $\mathbf{M}^I$. We have the corresponding conformal compactifications $\mathbf{P}$ and $\mathbf{F}$ such that $\mathbf{P}^I \subset \mathbf{P}$ and $\mathbf{F}^I \subset \mathbf{F}$, respectively.

A point in $\mathbf{P}$ is defined as a totally null m-plane ($\cong P^m(\mathbf{C})$) which is called α-plane in $\mathbf{M}$ according to Penrose's terminology. On the other hand, a point in $\mathbf{M}$ is regarded as the set ($\cong \mathbf{P}'$, $\dim \mathbf{P}' = m(m-1)/2$) of all totally null m-planes called α-planes through the point.

These manifolds are represented as homogeneous spaces in the following way.

The conformal compactification of $M^I = \mathbf{R}^n_1$ is the quotient space of $SO_0(n,2)$ by the minimal parabolic subgroup

$$M \cong S^{n-1} \times_{\mathbf{Z}_2} S^1 \cong SO_0(n,2)/SO_0(n-1,1)\cdot \mathbf{R}_+ \cdot \mathbf{R}^n,$$

where $SO_0(n,2)$ is the identity connected component of O(n,2).

The conformal compactification of the complexified $\mathbf{M}^I = \mathbf{C}^n$ is

$$\mathbf{M} \cong Q_n(\mathbf{C}) \cong SO(n+2)/SO(2)\times SO(n) \cong SO(n+2,\mathbf{C})/U,$$

where $Q_n(\mathbf{C})$ is a complex quadric and $SO(n+2,\mathbf{C})/U$ is a flag manifold. Remark that

$$\mathbf{M}^{\pm} \cong SO_0(n,2)/SO(2)\times SO(n)\ (\subset \mathbf{M}^I)$$

is a symmetric bounded of type IV. For $\mathbf{P}^I$,

$$\mathbf{P}_{m+1} := \mathbf{P} \cong SO(2m+2)/U(m+1) \cong SO(n+2,\mathbf{C})/Q.$$

Remark that

$$\mathbf{P}^{\pm} \cong SO^*(2m+2)/U(m+1)\ (\subset \mathbf{P}^I)$$

is a symmetric bounded domain of type II.

Moreover, we have

$$\mathbf{P}' = \mathbf{P}_{(m)} \cong SO(2m)/U(m).$$

The set $\mathcal{P}$ of all totally null m-planes through the origin has two connected components $\mathbf{P}_{(+)}$ and $\mathbf{P}_{(-)}$. Here $\mathbf{P}_{(+)}$ is the set $\mathbf{P}' = \mathbf{P}_{(m)}$ of all α-planes and $\mathbf{P}_{(-)}$ is the set of all β-planes. The space $\mathcal{P}$ is characterized by each of the following which is equivalent to each other.

(1) The space of all m-dimensional totally null subspaces (through the origin) in $\mathbf{C}^{2m}$.

(2) The space of all projective pure spinors on $\mathbf{C}^{2m}$.

(3) The space of all orthogonal complex structures (with respect to a positive definite inner product) on $\mathbf{R}^{2m}$.

See [7,9] in detail.

4. What have been mentioned above can be summarized in the following diagram:

$$\begin{array}{ccccc}
\mathbf{C}^m \ni (z_{m+i}) \longrightarrow (u_i, v_{ij}; z_{m+i}) & = & (z_i, z_{m+i}; v_{ij} & \in \mathbf{F}^I \longleftarrow (v_{ij}) \in \mathbf{P}' \\
\mu \swarrow & & \searrow \nu & \\
\mathbf{P}^I \ni (u_j, v_{ij}) & & (z_i, z_{m+i}) \in \mathbf{M}^I & \\
\downarrow\uparrow \iota & & k \uparrow\downarrow & \\
\mathbf{P}^I \ni (v_{ij}) & & (z_{m+i}) \in \mathbf{C}^m &
\end{array}$$

Here the mapping ι means the $n(=2m)$-dimensional family of sections with parameters (z_i, z_{m+i}). The mapping k means the $(m(m+1)/2)$-dimensional family of sections with parameters (u_i, v_{ij}).

2. Twistor integral representations of solutions of massless field equations

1. In the n-dimensional space–time $\mathbf{M}^I$, the equation of motion of a free scalar field ϕ with spin 0 and mass 0, i.e., massless Klein–Gordon equation is

$$\Box\phi = \sum_{i=1}^{m} \frac{\partial^2\phi}{\partial z_i \partial z_{m+i}} = 0.$$

This is a complex wave or Laplace equation. On the other hand, a free spinor field ψ with spin $\frac{1}{2}$ and mass 0 is represented by a massless Dirac equation (Weyl equation) $D\psi = 0$, see [3]. However, we do not discuss the latter.

As is well known, a field ϕ has a plane wave decomposition, in other words, a Fourier integral representation. This integral is integrated over the momentum space of the space–time. But each plane wave, which is represented as an exponential function, does not tend to 0 at the infinity.

We want to construct a field ϕ in the n-dimensional space–time $\mathbf{M}^I$ as an integral over the twistor space $\mathbf{P}^I$, i.e., as a twistor integral representation. Namely, we want to extend Penrose's twistor integral representation on the four-dimensional space–time to general dimensions.

2. We have the following proposition.

Proposition 2. *We define the function* $\phi(z)$ *by putting*

$$\phi(z) = \int\limits_{\Delta^k \subset \mathbf{P}'_z} f\left(z_i + \sum_{j=1}^{m} v_{ij} z_{m+j}, v_{ij}\right) \wedge^k \mathrm{d}v_{ij},$$

where $z = (z_i, z_{m+i})$, $u_i + \sum_{j=1}^{m} v_{ij} z_{m+j}$, Δ^k *is a* k*-chain in* $\mathbf{P}'_z = \nu^{-1}(z)$, $\wedge^k \mathrm{d}v_{ij}$ *is an exterior product* k*-form by any of* $\mathrm{d}v_{ij}$ *and* $f = f(u_i, v_{ij})$ *denotes a suitable analytic function on* $\mathbf{P}^I$ *such that* $f(z_i + \sum_{j=1}^{m} v_{ij} z_{m+j}, v_{ij})$ *is holomorphic in* Δ^k.

Then ϕ *satisfies*

$$\Box\phi = \sum_{i=1}^{m} \frac{\partial^2\phi}{\partial z_i \partial z_{m+i}} = 0.$$

Remark that a k-form $f \wedge^k \mathrm{d}v_{ij}$ on $\mathbf{P}'_z$ can be regarded as an element of $\Phi_z^* K \otimes \wedge^k T^*(\mathbf{P}')$, where Φ_z is a cross-section (a graph) of the bundle $\mathbf{P}^I \longrightarrow \mathbf{P}'$ defined by $u_i = z_i + \sum_{j=1}^{m} v_{ij} z_{m+j}$, and K is the field of meromorphic functions.

An elementary state is defined as $f = u_1^{\lambda_1} u_2^{\lambda_2} \cdots u_m^{\lambda_m}$ ($\Lambda_i \in \mathbf{C}$). For an analytic function $g = g(v_{ij})$ on $\mathbf{P}'$, an integral transformation

$$g \longmapsto \phi_g = \int\limits_{\Delta}^{k} g(v_{ij}) u_1^{\lambda_1} u_2^{\lambda_2} \cdots u_m^{\lambda_m} \wedge^k \mathrm{d}v_{ij}$$

is defined, if the integral exists. Since ϕ_g satisfies $\Box\phi_g = \sum_{i=1}^{m}(\partial^2\phi_g/\partial z_i\partial z_{m+i}) = 0$, it is a Poisson transformation too.

3. Differential forms associated with labeled trees and propagator functions

1. In this section we shall consider the twistor integral representation of the causal Green function (Feynman's propagator) of a massless Klein–Gordon equation in the n-dimensional space–time $\mathbf{M}^I$. This propagator function is the complex fundamental solution of a complex wave or Laplace equation. We show that this is given by the integration of $(m-1)$-forms associated with labeled trees. The real propagator function is obtained by the limit to M^I from imaginary regions of the difference between the retarded and the advanced Green functions. In this sense, it is a Sato's hyperfunction.

Let T be a graph with m vertices which has neither cycles nor loops, i.e., a tree with the following properties:

(1) $V(T)$ the set of vertices of T are labeled by the m numbers $\{1, 2, \ldots, m\}$.

(2) T is directed starting from the root 1. Each edge (joined by $i, j \in V(T)$) is oriented as $j \to i$. In this case, we call j the predecessor of i.

(3) The edges are ordered as

$$(p(2)2) < (p(3)3) < \cdots < (p(m)m), \quad (ij) = -(ji),$$

where $p(i)$ is the predecessor of i.

According to Cayley's theorem in graph theory, the number of labeled trees with m vertices up to isomorphism is m^{m-2}. See [6].

We let an edge (ij) and a vertex k correspond to $\mathrm{d}v_{ij}$ and u_k, respectively.

We define the $(m-1)$-form ω_T associated with T as follows:

$$\omega_T = \frac{(d_1-1)!(d_2-2)!\cdots(d_m-1)!}{(\frac{1}{2}(d_1+d_2+\cdots+d_m)-1)!}\cdot\frac{\mathrm{d}v_{p(2)2}\wedge\mathrm{d}v_{p(3)3}\wedge\cdots\wedge\mathrm{d}v_{p(m)m}}{u_1^{d_1}u_2^{d_2}\cdots u_m^{d_m}},$$

where d_i is the degree of i, i.e., the number of edges incident with i. Furthermore, we define an $(m-1)$-form ω as follows:

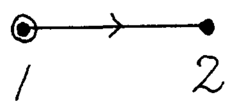
$$\omega = \sum_{T\in\mathcal{T}_m}\omega_T,$$

where $\mathcal{T}_m$ is the set of all labeled trees with m vertices up to isomorphism.

We give a few examples of ω_T, ω in low dimensions.

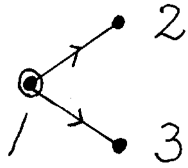
In the case where $n = 4$, i.e., $m = 2$, the number of $\mathcal{T}_2$ is $2^2 - 2 = 1$,

$$\omega_T = \frac{\mathrm{d}v_{12}}{u_1u_2} \quad \text{for } T:$$

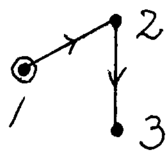


In the case where $n = 6$, i.e., $m = 3$, the number of T_3 is $3^{3-2} = 3$,

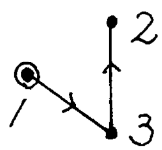
$$\omega_{T_1} = \frac{dv_{12} \wedge dv_{13}}{u_1^2 u_2 u_3} \quad \text{for } T_1 :$$



$$\omega_{T_2} = \frac{dv_{12} \wedge dv_{23}}{u_1 u_2^2 u_3} \quad \text{for } T_2 :$$

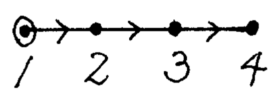


$$\omega_{T_3} = \frac{dv_{32} \wedge dv_{13}}{u_1 u_2 u_3^2} \quad \text{for } T_3 :$$

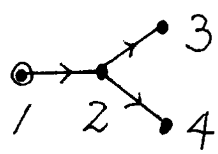


In the case where $n = 8$, i.e., $m = 4$, the number of $\mathcal{T}_4$ is $4^{4-2} = 16$, for instance,

$$\omega_{T_1} = \frac{1}{2} \cdot \frac{dv_{12} \wedge dv_{23} \wedge dv_{34}}{u_1 u_2^2 u_3^2 u_4} \quad \text{for } T_1 :$$

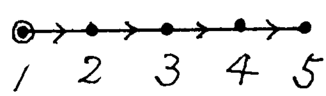


$$\omega_{T_2} = \frac{1}{2} \cdot \frac{dv_{12} \wedge dv_{23} \wedge dv_{24}}{u_1 u_2^3 u_3 u_4} \quad \text{for } T_2 :$$

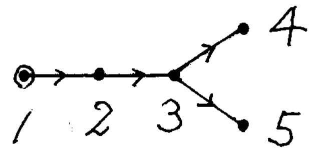


In the case where $n = 10$, i.e., $m = 5$, the number of $\mathcal{T}_5$ is $5^{5-2} = 125$, for instance,

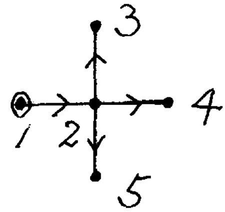
$$\omega_{T_1} = \frac{1}{6} \cdot \frac{dv_{12} \wedge dv_{23} \wedge dv_{34} \wedge dv_{45}}{u_1 u_2^2 u_3^2 u_4^2 u_5} \quad \text{for } T_1 :$$



$$\omega_{T_3} = \frac{1}{3} \cdot \frac{dv_{12} \wedge dv_{23} \wedge dv_{34} \wedge dv_{35}}{u_1 u_2^2 u_3^2 u_4 u_5} \quad \text{for } T_2 :$$



$$\omega_{T_3} = \frac{\mathrm{d}v_{12} \wedge \mathrm{d}v_{23} \wedge \mathrm{d}v_{24} \wedge \mathrm{d}v_{25}}{u_1 u_2^4 u_3 u_4 u_5} \quad \text{for } T_3:$$



2. We show that the $(m-1)$-form ω is a closed form on the fiber $\mathbf{P}'_z = \nu^{-1}(z)$ over $z \in \mathbf{M}^l$ with respect to the bundle $\mathbf{F}^l \longleftarrow \mathbf{M}^l$, or on the image $\Phi_z(\mathbf{P}')$ of a cross-section Φ_z with respect to the bundle $\mathbf{P}^l \longrightarrow \mathbf{P}'$.

Proposition 3. *We have*

$$\mathrm{d}\omega = 0 \quad \text{on } \mathbf{P}'_z \text{ (or } \Phi_z(\mathbf{P}')\text{)}.$$

Proof. We remark that, to the $(m-1)$-form associated with a labeled tree (here, we assume that the $(m-1)$-form does not include the part of $\mathrm{d}v_{ij}$), the left operation of the exterior product by a 1-form $\mathrm{d}v_{ij}$ corresponds to make a cycle by adding the edge from a vertex i to a vertex j for the labeled tree.

From now on, we simply write the 1-form $\mathrm{d}v_{ij}$ by (ij).

(1) We consider the following two labeled trees T, T' (Figs. 1 and 2).

Here $k \neq 1$, and $p(k) = i$ for T, $p(k) = j$ for T'. Moreover, as a graph with labeled vertices, the graph made by adding the edge from a vertex j to a vertex k for T and the graph made by adding the edge from a vertex i to a vertex k for T' are the same.

The $(m-1)$-forms ω_T, ω'_T associated with T, T' are as follows:

$$\omega_T = \frac{(d_1-1)!\cdots(d_i-1)!\cdots(d_m-1)!}{(\frac{1}{2}(d_1+\cdots+d_i+\cdots+d_j+\cdots+d_m)-1)!} \times \frac{(p(2)2)\wedge\cdots\wedge(ik)\wedge\cdots\wedge(p(m)m))}{u_1^{d_1}\cdots u_i^{d_i}\cdots u_j^{d_j}\cdots u_m^{d_m}},$$

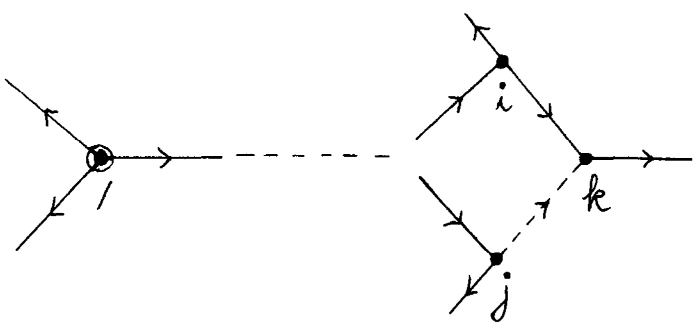


Fig. 1.

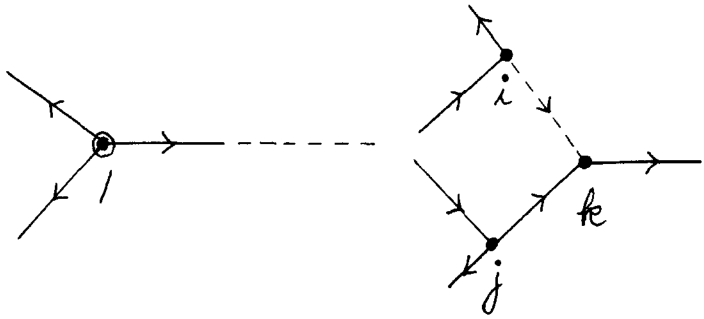


Fig. 2.

$$\omega'_T = \frac{(d_1-1)!\cdots(d_i-2)!\cdots(d_m-1)!}{(\frac{1}{2}(d_1+\cdots+(d_i-1)+\cdots+(d_j+1)+\cdots+d_m)-1)!} \times \frac{(p(2)2)\wedge\cdots\wedge(jk)\wedge\cdots\wedge(p(m)m)}{u_1^{d_1}\cdots u_i^{d_i-1}\cdots u_j^{d_j+1}\cdots u_m^{d_m}}$$

Taking the exterior derivations of ω_T, $\omega_{T'}$, we have

$$\begin{aligned}
\mathrm{d}\omega_T &= \frac{(d_1-1)!\cdots(d_i-1)!\cdots(d_j-1)(d_m-1)!}{(\frac{1}{2}(d_1+\cdots+d_m)-1)!} d(u_1^{-d_1}\cdots u_i^{-d_i}\cdots u_j^{-d_j}\cdots u_m^{-d_m}) \\
&\quad \wedge\cdots\wedge(p(m)m) \\
&= \frac{(d_1-1)!..(d_i-1)!..(d_j-1)(d_m-1)!}{(\frac{1}{2}(d_1+\cdots+d_m)-1)!} u_1^{-d_1}\cdots u_i^{-d_i}\cdots\overset{j}{\vee}\cdots u_m^{-d_m} d(u_j^{-d_j}) \\
&\quad \wedge\cdots\wedge(p(m)m)+\{\cdots\} \quad (\{\cdots\} \text{ is the term including differentials} \\
&\quad \text{except that of } u_j^{-d_j}) \\
&= -\frac{(d_1-1)!\cdots(d_i-1)!\cdots(d_j-1)(d_m-1)!}{(\frac{1}{2}(d_1+\cdots+d_m)-1)!} \\
&\quad \times \frac{z_{k+m}(jk)\wedge(p(2)2)\wedge\cdots\wedge(ik)\wedge\cdots\wedge(p(m)m)}{u_1^{d_1}\cdots u_i^{d_i}\cdots u_j^{d_j+1}\cdots u_m^{d_m}} + \{\cdots\}, \\
\mathrm{d}\omega_{T'} &= \frac{(d_1-1)!\cdots(d_i-2)!\cdots d_j!\cdots(d_m-1)!}{(\frac{1}{2}(d_1+\cdots+d_m)-1)!} \\
&\quad \times d(u_1^{-d_1}\cdots u_i^{-d_i+1}\cdots u_j^{-d^j-1}\cdots u_m^{-d_m})\wedge\cdots\wedge(p(m)m) \\
&= \frac{(d_1-1)!\cdots(d_i-2)!\cdots d_j!\cdots(d_m-1)!}{(\frac{1}{2}(d_1+\cdots+d_m)-1)!} \\
&\quad u_1^{-d_1}\cdots\overset{i}{\vee}\cdots u_j^{-d_j-1}\cdots u_m^{-d_m} d(u_i^{-d_i+1})\wedge\cdots\wedge(p(m)m) \\
&\quad +\{\cdots\} \quad (\{\cdots\} \text{ is the term including differentials except that of } u_i^{-d_i+1}) \\
&= -\frac{(d_1-1)!\cdots(d_i-1)!\cdots d_j\cdots(d_m-1)!}{(\frac{1}{2}(d_1+\cdots+d_m)-1)!}
\end{aligned}$$

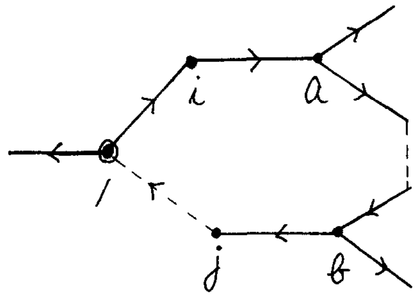


Fig. 3.

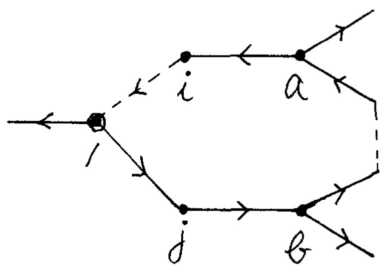


Fig. 4.

$$\times \frac{z_{k+m}(ik) \wedge (p(2)2) \wedge \cdots \wedge (jk) \wedge \cdots \wedge (p(m)m)}{u_1^{d_1} \cdots u_i^{d_i} \cdots u_j^{d_j+1} \cdots u_m^{d_m}} + \{\cdots\}.$$

Here, each third equality holds from the twistor equations. The coefficient of the first term of $\mathrm{d}\omega_T$ is equal to that of the first term of $\mathrm{d}\omega_{T'}$, and taking into consideration the equality

$$\begin{aligned}&(jk) \wedge (p(2)2) \wedge \cdots \wedge (ik) \wedge \cdots \wedge (p(m)m)\\ &\quad = -(ik) \wedge (p(2)2) \wedge \cdots \wedge (jk) \wedge \cdots \wedge (p(m)m),\end{aligned}$$

it follows that the first term of $\mathrm{d}\omega_{T'}$ added to that of $\mathrm{d}\omega_T$ makes zero.

(2) We consider the following two labeled trees T, T' (Figs. 3 and 4). Here $i < j$, and $p(i) = 1$ for T, $p(j) = 1$ for T'. Moreover, as a graph with labeled vertices (and nondirected edges), the graph made by adding the edge from a vertex j to a vertex 1 for T and the graph made by adding the edge from i to a vertex 1 for T' are the same, and the path with the length r, $1 \to i \to a \to \cdots \to b \to j \to 1$, is a cycle.

The $m-1$-forms ω_T, ω_T' associated with T, T' are as follows:

$$\begin{aligned}\omega_T &= A(p(2)2) \wedge \cdots \wedge (1i) \wedge \cdots \wedge (ia) \wedge \cdots \wedge (p(b)b)\\ &\quad \wedge \cdots \wedge (bj) \wedge \cdots \wedge (p(m)m),\\ \omega_{T'} &= A'(p(2)2) \wedge \cdots \wedge (ai) \wedge \cdots \wedge (p(a)a) \wedge \cdots \wedge (jb)\\ &\quad \wedge \cdots \wedge (1j) \wedge \cdots \wedge (p(m)m),\end{aligned}$$

where, A and A' are the coefficients of ω_T and $\omega_{T'}$, respectively, and $p(j) = b$ for T, $p(i) = a$ for T'. Taking the exterior derivatives of ω_T and $\omega_{T'}$, we have

$$\begin{aligned} \mathrm{d}\omega_T &= B(j1) \wedge (p(2)2) \wedge \cdots \wedge (1i) \wedge \cdots \wedge (bj) \wedge \cdots \wedge (p(m)m) + \{\cdots\}, \\ \mathrm{d}\omega_{T'} &= B'(i1) \wedge (p(2)2) \wedge \cdots \wedge (ai) \wedge \cdots \wedge (1j) \wedge \cdots \wedge (p(m)m) + \{\cdots\}. \end{aligned}$$

Here the calculation is performed in the same way as (1). It follows that $B = B'$ holds. We show that the first term of $\mathrm{d}\omega_{T'}$ added to that of $\mathrm{d}\omega_T$ makes zero.

If we change the root 1 to a new root i, then the direction of an only edge $(1i)$ for T is changed and the directions of the only edges of a path $P : 1 \to j \to b \to \cdots \to a \to i$ for T' are changed. We rewrite ω_T and $\omega_{T'}$ as if the root is i. As $(1i) = -(i1)$ for ω_T,

$$\begin{aligned} \omega_T = -A(p(2)2) \wedge \cdots \wedge (i1) \wedge \cdots \wedge (ia) \wedge \cdots \wedge (p(b)b) \\ \wedge \cdots \wedge (bj) \wedge \cdots \wedge (p(m)m). \end{aligned}$$

As $(ai) = -(ia), \ldots, (p(a)a) = -(ap(a)), \ldots, (jb) = -(bj), \ldots, (1j) = -(j1)$ for $\omega_{T'}$,

$$\begin{aligned} \omega_{T'} = (-1)^{r-1}A'(p(2)2) \wedge \cdots \wedge (ia) \wedge \cdots \wedge (ap(a)) \wedge \cdots \wedge (bj) \\ \wedge \cdots \wedge (j1) \wedge \cdots \wedge (p(m)m). \end{aligned}$$

We let a wedge $(j1)$ head the wedges of P: $(ia), \ldots, (ap(a)), \ldots, (bj), \ldots, (j1)$. That is, $(j1), (ia), \ldots, (ap(a)), \ldots, (bj), \ldots$. This is made by r permutations. Rearranging $\omega_{T'}$, we have

$$\begin{aligned} \omega_{T'} = A'(p(2)2) \wedge \cdots \wedge (j1) \wedge \cdots \wedge (ia) \wedge \cdots \wedge (ap(a)) \\ \wedge \cdots \wedge (bj) \wedge \cdots \wedge (p(m)m). \end{aligned}$$

The part $(i1)$ of ω_T and the part $(j1)$ of $\omega_{T'}$ are the same places by counting from the parts $(p(2)2)$. The other parts of ω_T and $\omega_{T'}$ are the same. Therefore, since

$$\begin{aligned} \mathrm{d}\omega_T &= -B(ji) \wedge (p(2)2) \wedge \cdots \wedge (i1) \wedge \cdots \wedge (p(m)m) + \{\cdots\}, \\ \mathrm{d}\omega_{T'} &= -B(i1) \wedge (p(2)2) \wedge \cdots \wedge (j1) \wedge \cdots \wedge (p(m)m) + \{\cdots\}, \end{aligned}$$

it follows that the first term of $\mathrm{d}\omega_{T'}$ added to that of $\mathrm{d}\omega_T$ makes zero.

Let a labeled tree T be given. Then a graph with a cycle by adding an edge incident with a vertex k to T cancels out the graph with a cycle by adding an edge incident with the same vertex k to another lebeled tree T' which is unique determined. Summing up the forms corresponding to all labeled trees, we conclude that $\mathrm{d}\omega = 0$. □

From Proposition 3, the $(m-1)$-form ω determines a cohomology class $[\omega]$ of degree $(m-1)$ on a hyperplane configuration complement $\mathbf{P}'_z - \bigcup_{i=1}^m H_i$. Here H_i is a hyperplane in $\mathbf{P}'_z$ defined by a linear equation $u_i = 0$. Remark that $\mathbf{P}'_z \cong \mathbf{C}^{m(m-1)/2}$. Hence we have the following.

Corollary 1. *The class* $[\omega]$ *belongs to* $H^{m-1}(\mathbf{P}'_z - \bigcup_{i=1}^m H_i, \mathbf{C})$.

In 4 in this section, we will represent the class $[\omega]$ as linear combination of logarithmic forms in $\mathbf{P}'_z - \bigcup_{i=1}^m H_i$.

3. We show that the propagator function $\left(\sum_{i=1}^m z_i z_{m+i}\right)^{-m+1}$ on $\mathbf{M}^I$ is given by the integration of the $(m-1)$-form ω over an appropriate $(m-1)$-cycle Δ in $\mathbf{P}'_z$.

Theorem 1. *We have*

$$\int_\Delta \omega = (2\pi i)^{m-1}\left(\sum_{i=1}^m z_i z_{m+i}\right)^{-m+1},$$

where $\Delta = \sum_{i=1}^m c_i \Delta_i$, $\sum_{i=1}^m c_i = 1$ *and* Δ_i *is an* $(m-1)$*-cycle transversal to* $\bigcap_{j\neq i} H_j$ *in* $\mathbf{P}'_z$.

Proof. We have

$$\int_\Delta \omega = \int_{\sum_i c_i \Delta_i} \omega = \sum_i c_i \int_{\Delta_i} \omega.$$

We show that

$$\int_{\Delta_i} \omega = (2\pi i)^{m-1}\left(\sum_{i=1}^m z_i z_{m+i}\right)^{-m+1},$$

for each i. Take $\Delta = \Delta_m$. It is an $(m-1)$-cycle transversal to a subspace defined by $u_i = \cdots = u_{m-1} = 0$ and we take it so that it may avoid a hypersurface defined by $u_m = 0$. Furthermore, we restrict the $(m-1)$-cycle Δ_m to a homologous one on a subspace defined by $v_{ij} = 0$ $(i, j \neq m)$. Then, by residue theorem (cf. [4]),

$$\begin{aligned}
\int_{\Delta_m} \omega &= (2\pi i)^{m-1}\,\mathrm{Res}_{u1=\cdots=u_{m-1}=0}\,\omega|_{v_{ij}=0(i,j\neq m)} \\
&= (2\pi i)^{m-1}\,\mathrm{Res}\frac{dv_{m2}\wedge dv_{m3}\wedge\cdots\wedge dv_{mm-1}\wedge dv_{1m}}{u_1u_2\cdots u_{m-1}u_m^{m-1}} \\
&= (2\pi i)^{m-1}\,\mathrm{Res}\frac{dv_{1m}\wedge dv_{2m}\wedge\cdots\wedge dv_{m-1m}}{u_1u_2\cdots u_{m-1}u_m^{m-1}} \\
&= (2\pi i)^{m-1} \\
&\quad\times\frac{1}{z_{2m}^{m-1}(z_m+(z_1/z_{2m})z_{m+1}+(z_2/z_{2m})z_{m+2}+\cdots+(z_{m-1}/z_{2m})z_{2m-1})^{m-1}} \\
&= (2\pi i)^{m-1}\frac{1}{(z_1z_{m+1}+z_2z_{m+2}+\cdots+z_mz_{2m})^{m-1}}.
\end{aligned}$$

Here, the second equality holds from $dv_{ij} = 0$ $(i, j \neq m)$. It means that the integration of the $(m-1)$-form ω associated to all m^{m-2} labeled trees is reduced to the integration of ω_T associated to the following labeled tree T (Fig. 5).

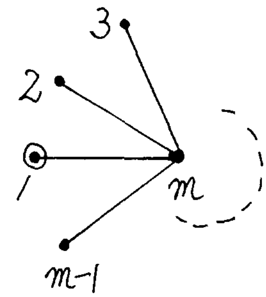


Fig. 5.

For the fourth equality, it follows from

$$\left|\frac{\partial(u_1, u_2, \ldots, u_{m-1})}{\partial(v_{1m}, v_{2m}, \ldots, v_{m-1m})}\right| = z_{2m}^{m-1}$$

and

$$v_{im} = -v_{mi} = -\frac{z_i}{z_{2m}}.$$

Similarly we obtain it for $\Delta = \Delta_i$ $(i \neq m)$. $\square$

4. Let X be a hyperplane configuration complement in $\mathbf{C}^n$, i.e.,

$$X = \mathbf{C}^n - \bigcup_j H_j,$$

where H_j is a hyperplane in $\mathbf{C}^n$ defined by a linear equation $f_j = 0$. Then, by Brieskorn and others [11], the holomorphic de Rham cohomology $H^*(X, \mathbf{C})$ is isomorphic to Orlik–Solomon algebra:

$$H^*(X, \mathbf{C}) \cong \sum_{p=0}^{n} \sum_{i_1 < \cdots < i_p} \mathbf{C}(\mathrm{d}\log f_{i_1} \wedge \cdots \wedge \mathrm{d}\log f_{i_p}).$$

Here logarithmic forms $\{\mathrm{d}\log f_j\}$ are the generator of the algebra.

In our case, it follows that

$$H^{m-1}\left(\mathbf{P}_z' - \bigcup_{i=1}^{m} H_i, \mathbf{C}\right) \cong \mathbf{C}^m$$

and the basis is

$$\{\mathrm{d}\log u_1 \wedge \cdots \overset{i}{\vee} \cdots \wedge \mathrm{d}\log u_m | i = 1, \ldots, m\}.$$

Proposition 4. *As the cohomology class, the $(m-1)$-form ω is written by the following linear combination of the above basis:*

$$\omega \sim \left(\sum_{i=1}^{m} z_i z_{m+1}\right)^{-m+1} \left(\sum_{j=1}^{m} (-1)^{j-1} \,\mathrm{d}\log u_1 \wedge \cdots \overset{j}{\vee} \cdots \wedge \,\mathrm{d}\log u_m\right)$$

in $H^{m-1}(\mathbf{P}'_z - \bigcup_{i=1}^{m} H_i, \mathbf{C})$.

Proof. Put

$$\omega = \sum_{j=1}^{m} a_j (-1)^{j-1} \,\mathrm{d}\log u_1 \wedge \cdots \overset{j}{\vee} \cdots \wedge \,\mathrm{d}\log u_m.$$

Then we have

$$\int_{\Delta_j} \omega = a_j (2\pi i)^{m-1}$$

for j. Since

$$\int_{\Delta_j} \omega = (2\pi i)^{m-1} \left(\sum_{i=1}^{m} z_i z_{m+i}\right)^{-m+1},$$

it follows that

$$a_j = \left(\sum_{i=1}^{m} z_i z_{m+i}\right)^{-m+1}.$$

Therefore we have proved the claim. □

4. Odd dimensional cases

1. In this section we shall study twistor theory of the odd dimensional flat space–times. By the reduction to dimension $n + 1$ $(= 2m)$ (the method of Hadamard's descent), we consider the same argument as from Section 1 to Section 3.

We suppose that the dimension of $\mathbf{M}^I$ are odd dimensional, say $n = 2m - 1$.

We take the coordinate system $(z_i, z_{m+i}, z) = (z_1, \ldots, z_{m-1}, z_{m+1}, \ldots, z_{2m-1}, z)$ on $\mathbf{M}^I$ such that the complex metric $\mathrm{d}s^2$ is represented in the form

$$\mathrm{d}s^2 = 2\left(\mathrm{d}z^2 + \sum_{i=1}^{m-1} \mathrm{d}z_i \,\mathrm{d}z_{m+i}\right).$$

Let $\mathbf{P}^I$ be the set of all (degenerate) null m-planes in $\mathbf{M}^I$ including a (maximal) totally null $(m-1)$-plane. In odd dimensional cases we call $\mathbf{P}^I$ the twistor space of $\mathbf{M}^I$.

The twistor equations are obtained by the reduction to dimension $n+1$ $(= 2m)$. In $(n+1)$ $(= 2m)$-dimensional case, $\mathbf{P}^I$ is the set of all (maximal) totally null m-planes in $\mathbf{M}^I$. The twistor equations are

$$u_i = z_i + \sum_{j=1}^{m} v_{ij} z_{m+j}, \quad v_{ij} = -v_{ji}, \quad (i = 1, \dots, m).$$

In n ($= 2m - 1$)-dimensional case, we divide the above equations into the following two parts:

$$u_i = z_i + \sum_{j=1}^{m-1} v_{ij} z_{m+j} + v_{im} z_{2m} \quad (i = 1, \dots, m-1),$$

$$u_m = z_m + \sum_{j=1}^{m-1} v_{mj} z_{m+j},$$

and then, we define the following equations:

$$\begin{aligned} w_i &= u_i + v_{im} u_m \quad (i = 1, \dots, m-1) \\ &= z_i + v_{im}(z_m + z_{2m}) + \sum_{j=1}^{m-1} (v_{ij} - v_{im} v_{jm}) z_{m+j}. \end{aligned}$$

Putting

$$2z = z_m + z_{2m},$$

we get the following proposition.

Proposition 5. *A generic element belonging to* $\mathbf{P}^I$ *is represented by the following twistor equations:*

$$w_i = z_i + 2 v_{im} z + \sum_{j=1}^{m-1} (v_{ij} - v_{im} v_{jm}) z_{m+j} \quad (i = 1, \dots, m-1),$$

where $w_i, v_{ij} \in \mathbf{C}$, $v_{ij} = -v_{ji}$.

We remark that the twistor equations are linear with respect to z_i, z_{m+i}, z, but on the other hand they are quadratic with respect to v_{ij} unlike even dimensional cases. Compared with Penrose's four-dimensional twistor theory, Hitchin's three-dimensional minitwistor theory is so. See [8].

What have been mentioned above can be summarized in the following diagram:

$$\begin{array}{ccccc} \mathbf{C}^m \ni (z_{m+i}, z) \longrightarrow & (w_i, v_{ij}; z_{m+i}, z) = (z_i, z_{m+i}, z; v_{ij}) \in \mathbf{F}^I & \longleftarrow (v_{ij}) \in \mathbf{P}' \\ & \mu \swarrow \qquad\qquad \searrow \nu & \\ \mathbf{P}^I \ni (w_j, v_{ij}) & & (z_i, z_{m+i}, z) \in \mathbf{M}^I \\ \downarrow & & \downarrow \\ \mathbf{P}^I \ni (v_{ij}) & & (z_{m+i}, z) \in \mathbf{C}^m \end{array}$$

We remark that $\mathbf{P}'$ in n ($= 2m - 1$)-dimensional case is the same $\mathbf{P}'$ as in $(n+1)$ ($= 2m$)-dimensional case.

2. In the n $(= 2m - 1)$-dimensional space–time $\mathbf{M}^I$, the equation of motion of a free scalar field ϕ with spin 0 and mass 0, i.e., a massless Klein–Gordon equation is

$$\Box\phi = \left(\frac{1}{4}\frac{\partial^2}{\partial z^2} + \sum_{i=1}^{m-1}\frac{\partial^2}{\partial z_i \partial z_{m+i}}\right)\phi = 0.$$

The field ϕ has a twistor integral representation similar to even-dimensional cases. As in Proposition 2 in Section 2, we have the following proposition.

Proposition 6. *Let $f = f(w_i, v_{ij})$ be a suitable analytic function on $\mathbf{P}^I$. Put*

$$\phi(\mathbf{z}) = \int_{\Delta^k \subset \mathbf{P}'_z} f\left(z_i + 2v_{im}z + \sum_{j=1}^{m-1}(v_{ij} - v_{im}v_{jm})z_{m+j}, v_{ij}\right) \wedge^k \mathrm{d}v_{ij},$$

where $\mathbf{z} = (z_i, z_{m+i}, z)$, $w_i = z_i + 2v_{im}z + \sum_{j=1}^{m-1}(v_{ij} - v_{im}v_{jm})z_{m+j}$, Δ^k is a k-chain in $\mathbf{P}'_z = \nu^{-1}(\mathbf{z})$, $\wedge^k \mathrm{d}v_{ij}$ is a k-form.

Then ϕ satisfies

$$\Box\phi = \left(\frac{1}{4}\frac{\partial^2}{\partial z^2} + \sum_{i=1}^{m-1}\frac{\partial^2}{\partial z_i \partial z_{m+i}}\right)\phi = 0.$$

3. We shall consider the twistor integral representation of the propagator function of a massless Klein–Gordon equation in the n $(= 2m - 1)$-dimensional space–time $\mathbf{M}^I$ from the same point of view as in even dimensional space–times.

We take a labeled tree T with m vertices and furthermore the set $\mathcal{T}_m$ of all m^{m-2} labeled trees with m vertices mentioned in the $(n + 1)$ $(= 2m)$-dimensional case in Section 3.

In the $(n + 1)$ $(= 2m)$-dimensional case, we defined the $(m - 1)$-form ω_T associated with T as follows:

$$\omega_T = \frac{(d_1 - 1)! \cdots (d_m - 1)!}{(\frac{1}{2}(d_1 + \cdots + d_m) - 1)!}\frac{\mathrm{d}v_{p(2)2} \wedge \cdots \wedge \mathrm{d}v_{p(m)m}}{u_1^{d_1} \cdots u_m^{d_m}}.$$

In this n $(= 2m - 1)$-dimensional case, we define the $(m - 1)$-form θ_T associated with T as follows:

$$\theta_T = \sum_{e_1 + \cdots + e_{m-1} = d_{m-1}} \frac{(d_m - 1)!(d_1 + e_1 - 1)! \cdots (d_{m-1} + e_{m-1} - 1)!}{(\frac{1}{2}(d_1 + \cdots + d_m) - 1)! e_1! \cdots e_{m-1}!}$$
$$\frac{v_{1m}^{e_1} \cdots v_{m-1m}^{e_{m-1}} \mathrm{d}v_{p(1)1}1 \wedge \cdots \wedge \mathrm{d}v_{p(m-1)m-1}}{w_1^{d_1+e_1} \cdots w_{m-1}^{d_{m-1}+e_{m-1}}},$$

where we change a root of T to vertex m. Remark that the number of the term of $\sum$ is ${}_{m-1}H_{d_m-1} = \frac{(m+d_m-3)!}{d_{m-1}!(m-2)!}$.

Furthermore, we define an $(m - 1)$-form θ as follows:

$$\theta = \sum_{T \in \mathcal{T}_m} \theta_T.$$

We show that the $(m-1)$-form θ is closed on the fiber $\mathbf{P_z}' = \nu^{-1}(\mathbf{z})$ over $\mathbf{z} \in \mathbf{M}^l$ with respect to the bundle $\mathbf{F}^l \to \mathbf{M}^l$, or on the image $\Phi_{\mathbf{z}}(\mathbf{P}')$ of a cross-section $\Phi_{\mathbf{z}}$ with respect to the bundle $\mathbf{P}^l \to \mathbf{P}'$.

Proposition 7. *We have*

$$\mathrm{d}\theta = 0 \quad \text{on } \mathbf{P_z}' \quad (\text{or } \Phi_{\mathbf{z}}(\mathbf{P}')).$$

Proof.

Step 1. By Theorem 1 in $(n+1)$ $(=2m)$-dimensional case in Section 3 (we use the same notations),

$$\Box \int_{\Delta} \omega = (2\pi i)^{m-1} \left(\sum_{i=1}^{m} \frac{\partial^2}{\partial z_i \partial z_{m+i}} \left(\sum_{i=1}^{m} z_i z_{m+i} \right)^{-m+1} \right) = 0.$$

Put

$$2z = z_m + z_{2m}, \qquad 2w = z_m - z_{2m},$$

then

$$z_m z_{2m} = z^2 - w^2, \qquad \frac{\partial^2}{\partial z_m \partial z_{2m}} = \frac{1}{4} \left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial w^2} \right)$$

hold. Put

$$\Box = \sum_{i=1}^{m} \frac{\partial^2}{\partial z_i \partial z_{m+i}} = \sum_{i=1}^{m-1} \frac{\partial^2}{\partial z_i \partial z_{m+i}} + \frac{1}{4}\frac{\partial^2}{\partial z^2} - \frac{1}{4}\frac{\partial^2}{\partial w^2} = \Box' - \frac{1}{4}\frac{\partial^2}{\partial w^2},$$

where $\Box'$ is $\Box$ of dimension n $(=2m-1)$ in this section. Take a neighborhood $U \subset \mathbf{M}^l$ of $\mathbf{z}$ locally. In consideration of a product set $\Delta \times U$ for $\Delta \subset \nu^{-1}(\mathbf{z})$, we have

$$\Box \int_{\Delta} \omega = \int_{\Delta} \Box' \omega \int_{\Delta} \Box' \omega - \frac{1}{4} \int_{\Delta} \frac{\partial^2}{\partial w^2} \omega = 0.$$

Next, we integrate it over an appropriate cycle γ on a w plane:

$$\frac{1}{2\pi i} \int_{\gamma} \left(\int_{\Delta} \Box \omega \right) \mathrm{d}w = \frac{1}{2\pi i} \int_{\gamma} \left(\int_{\Delta} \Box' \omega \right) \mathrm{d}w - \frac{1}{4} \frac{1}{2\pi i} \int_{\gamma} \left(\int_{\Delta} \frac{\partial^2}{\partial w^2} \omega \right) \mathrm{d}w = 0.$$

We show that the second term is equal to 0. Then, it follows that the first term is equal to 0. By the way,

$$d_{v,w} \left(\frac{\partial}{\partial w} \omega \right) = \mathrm{d}w \wedge \frac{\partial^2}{\partial w^2} \omega,$$

where $d_{v,w}$ is the exterior derivative with respect to variables v_{ij}, w, and $d_v w = 0$ holds. Therefore, for the cycle $\Gamma = \Gamma(v, w)$ made by γ and Δ,

$$\int_\gamma \left(\int_\delta \frac{\partial^2}{\partial w^2} \omega \right) \mathrm{d}w = - \int_\Gamma \mathrm{d}w \wedge \frac{\partial^2}{\partial w^2} \omega = - \int_\Gamma d_{v,w} \left(\frac{\partial}{\partial w} \omega \right) = 0.$$

Now, let us put

$$\theta = \frac{1}{2\pi i} \int_\gamma \omega \, \mathrm{d}w,$$

then it follows that

$$d_v \theta = \frac{1}{2\pi i} \int_\gamma \mathrm{d}_v \omega \, \mathrm{d}w = 0.$$

For an appropriate cycle Δ' homologous to Δ,

$$\Box' \int_\Delta \theta = \Box' \int_{\Delta'} \frac{1}{2\pi i} \int_\gamma \omega \, \mathrm{d}w = \frac{1}{2\pi i} \int_\gamma \left(\int_{\Delta'} \Box' \omega \right) \mathrm{d}w = 0.$$

This θ is nothing but what we desire. In Step 2, we explicitly calculate θ.

Step 2. We solve $u_m = 0$ and take the residue.

From

$$u_m = z_m + \sum_{j=1}^{m-1} v_{mj} z_{m+j} = (z + w) + \sum_{j=1}^{m-1} v_{mj} z_{m+j} = 0,$$

it follows that

$$w = w_0 = -z - \sum_{j=1}^{m-1} v_{mj} z_{m+j}.$$

Let γ be a cycle round the point w_0 on a w plane. Recall that

$$\omega_T = \frac{(d_1 - 1)! \cdots (d_m - 1)!}{(\frac{1}{2}(d_1 + \cdots + d_m) - 1)!} u_1^{-d_1} \cdots u_{m-1}^{-d_{m-1}} u_m^{-d_m} \, \mathrm{d}v_{p(2)2} \wedge \cdots \wedge \mathrm{d}v_{p(m)m}.$$

Then

$$\begin{aligned} \theta_T &= \frac{1}{2\pi \mathrm{i}} \int_\gamma \omega_T \, \mathrm{d}w \\ &= \frac{1}{2\pi \mathrm{i}} 2\pi \mathrm{i} \, \mathrm{Res}_{w_0} (\omega_T \, \mathrm{d}w) \\ &= \frac{(d_1 - 1)! \cdots (d_m - 1)!}{(\frac{1}{2}(d_1 + \cdots + d_m) - 1)!} \frac{1}{(d_m - 1)!} \end{aligned}$$

$$\times \lim_{w \to w_0} \left(\frac{\mathrm{d}}{\mathrm{d}w}\right)^d_{m-1} (u_1^{-d_1} \cdots u_{m-1}^{-d_{m-1}} \mathrm{d}v_{p(2)2} \wedge \cdots \wedge \mathrm{d}v_{p(m)m}).$$

From

$$\begin{aligned} u_i &= z_i + \sum_{j=1}^{m-1} v_{ij} z_{m+j} + v_{im} z_{2m} \quad (i = 1, \ldots, m-1) \\ &= z_i \sum_{j=1}^{m-1} v_{ij} z_{m+j} + v_{im}(z - w), \end{aligned}$$

according to a long calculation by Leibniz' rule, etc., and the consequence that, as $w \to w_0$,

$$\begin{aligned} u_i &\to z_i + \sum_{j=1}^{m-1} v_{ij} z_{m+j} + v_{im} \left(2z + \sum_{j=1}^{m-1} v_{mj}^{m+j}\right) \\ &= z_i + 2v_{im} z + \sum_{j=1}^{m-1} (v_{ij} - v_{im} v_{jm}) z_{m+j} = w_i, \end{aligned}$$

we obtain

$$\begin{aligned} \theta_T = &\sum_{e_1 + \cdots + e_{m-1} = d_{m-1}} \frac{(d_m - 1)!(d_1 + e_1 - 1)! \cdots (d_{m-1} + e_{m-1} - 1)!}{(\frac{1}{2}(d_1 + \cdots + d_m) - 1)! e_1! \cdots e_{m-1}!} \\ &\times \frac{v_{1m}^{e_1} \cdots v_{m-1m}^{e_{m-1}} \mathrm{d}v_{p(1)1} \wedge \cdots \wedge \mathrm{d}v_{p(m-1)m-1}}{w_1^{d_1 + e_1} \cdots w_{m-1}^{d_{m-1} + e_{m-1}}}. \qquad \square \end{aligned}$$

4. We show that the propagator function $(z^2 + \sum_{i=1}^{m-1} z_i z_{m+1})^{-m+3/2}$ on $\mathbf{M}^I$ is given by the integration of the $(m-1)$-from θ over an appropriate $(m-1)$-cycle of $\mathbf{P}'_{\mathbf{z}}$.

Theorem 2. *We have*

$$\int_\Delta \theta = c \left(z^2 + \sum_{i=1}^{m-1} z_1 z_{m+i}\right)^{-m+3/2},$$

where Δ is an $(m-1)$-cycle transversal to $\bigcap_i G_i$ $(G_i : w_i = 0)$ in $\mathbf{P}'_{\mathbf{z}}$, and c is some constant.

Proof. We restrict the $(m-1)$-cycle Δ to a homologous one on a subspace defined by $v_{ij} = 0$ $(i, j \neq 1)$. For

$$w_i = u_i + v_{im} u_m = 0 \quad (i = 2, \ldots, m-1),$$

from $v_{im} = 0$, it follows that

$$v_{12} = \frac{z_2}{z_{m+1}}, \ldots, v_{1m-1} = \frac{z_{m-1}}{z_{m+1}},$$

and for $w_1 = 0$, that is to say,

$$\begin{aligned} w_1 &= z_1 + 2v_{1m}z + \sum_{j=1}^{m-1}(v_{1j} - v_{1m}v_{jm})z_{m+j} \\ &= z_1 + 2v_{1m}z + \sum_{j=1}^{m-1} v_{ij}z_{m+j} - v_{1m}^2 z_{m+1} \\ &= z_1 + \frac{z_2}{z_{m+1}}z_{m+2} + \cdots + \frac{z_{m-1}}{z_{m+1}}z_{2m-1} + 2zv_{1m} - z_{m+1}v_{1m}^2 = 0, \end{aligned}$$

we regard it as an algebraic equation of degree 2 with respect to v_{1m}. Let α and β be its solutions: $-z_{m+1}(v_{1m} - \alpha)(v_{1m} - \beta) = 0$. Since the discriminant D is

$$\begin{aligned} D &= z^2 + \frac{z_1 z_{m+1} + \cdots + z_{m-1}z_{2m-1}}{z_{m+1}} z_{m+1} \\ &= z^2 + z_1 z_{m+1} + \cdots + z_{m-1}z_{2m-1}, \end{aligned}$$

if follows that

$$v_{1m} = \alpha, \qquad \beta = \frac{1}{z_{m+1}}(z + \sqrt{D}).$$

By residue theorem,

$$\begin{aligned} \int_\Delta \theta &= (2\pi \mathrm{i})^{m-1} \operatorname{Res}_{w_1=\cdots=w_{m-1}=0} \theta|_{v_{ij}=0\ (i,j\neq 1)} \\ &= (2\pi \mathrm{i})^{m-1} \operatorname{Res} \frac{\mathrm{d}v_{12} \wedge \cdots \wedge \mathrm{d}v_{1m-1} \wedge \mathrm{d}v_{1m}}{w_1^{m-1} w_2 \cdots w_{m-1}} \\ &= (2\pi \mathrm{i})^{m-1} \frac{(-1)^{m-1}}{z_{m+1}^{m-2}} \operatorname{Res}_{v_{1m}} = \alpha \frac{\mathrm{d}v_{1m}}{w_1^{m-1}} \\ &= (2\pi \mathrm{i})^{m-1} \frac{(-1)^{m-1}}{z_{m+1}^{m-2}} \int_\delta \frac{\mathrm{d}v_{1m}}{(-z_{m+1})^{m-1}(v_{1m} - \alpha)^{m-1}(v_{1m} - \beta)^{m-1}} \\ &= (2\pi \mathrm{i})^{m-1} \frac{(-1)^{m-1}}{z_{m+1}^{m-2}} \frac{1}{(-1)^{m-1} z_{m+1}^{m-1}} \frac{1}{(\alpha - \beta)^{2m-3}} \\ &\quad \times \frac{(m-1)m \cdots (2m-4)(-1)^{m-2}}{(m-2)!} \\ &= (2\pi \mathrm{i})^{m-1} \frac{1}{z_{m+1}^{2m-3}} \frac{1}{(2\sqrt{D}/z_{m+1})^{2m-3}} (-1)^{m-2} \binom{2m-4}{m-2} \\ &= (2\pi \mathrm{i})^{m-1} \frac{1}{D^{m-3/2}} (-1)^{m-2} 2^{-2m+3} \binom{2m-4}{m-2} \\ &= c \frac{1}{(z^2 + \sum_{i=1}^{m-1} z_i z_{m+i})^{m-3/2}}, \end{aligned}$$

where δ is a cycle round the point $v_{1m} = \alpha$ on a v_{1m} plane and

$$c = (2\pi \mathrm{i})^{m-1}(-1)^{m-2}2^{-2m+3}\binom{2m-4}{m-2}.$$

The third follows from

$$\left|\frac{\partial(w_2, \ldots, w_{m-1})}{\partial(v_{12}, \ldots, v_{1m-1})}\right| = (-1)^{m-2} z_{m+1}^{m-2},$$

and the sixth equality follows from the relation $\alpha - \beta = 2\sqrt{D}/z_{m+1}$. □

References

[1] M.F. Atiyah, Green's functions for self-dual four-manifolds, Mathematical Analysis and Applications, Part A, Advances in Mathematics, Supplement Studies, vol. 7A, Academic Press, New York, 1981, pp. 129–158.

[2] R.J. Baston, M.G. Eastwood, The Penrose Transform, Oxford University Press, Oxford, 1989.

[3] R. Delanghe, F. Sommen and V. Souček, Clifford Algebra and Spinor-Valued Functions, Kluwer Academic Publishers, Dordrecht, 1992.

[4] P. Griffiths, J. Harris, Principles of Algebraic Geometry, Wiley/Interscience, New York, 1978.

[5] E.L. Grinberg, Euclidean Radon transforms: ranges and restrictions, Contemporary Math. 63 (1987) 109–133.

[6] F. Harary, Graph Theory, Addison-Wesley, Reading, MA, 1969.

[7] F.R. Harvey, Spinor and Calibrations, Academic Press, Boston, 1990.

[8] N.J. Hitchin, in: H.D. Doebner, T.D. Palev (Eds.), Complex Manifolds and Einstein's Equations, Twistor Geometry and Nonlinear Systems, Lecture Notes in Mathematics, vol. 907, Springer, Berlin, 1982, pp. 73–99.

[9] H.B. Lawson Jr., M. Michelson, Spin Geometry, Princeton University Press, Princeton, NJ, 1989.

[10] Y.I. Manin, Gauge Field Theory and Complex Geometry, Spinger, Berlin, 1984.

[11] P. Orlik, H. Terao, Arrangements of Hyperplanes, Spinger, Berlin, Heidelberg, New York, 1992.

[12] R. Penrose, W. Rindler, Spinors and Space–Time, vol. 2, Cambridge University Press, Cambridge, 1986.

[13] R.S. Ward, R.O. Wells Jr., Twistor Geometry and Field Theory, Cambridge University Press, Cambridge, 1990.

[14] R.O. Wells Jr., Complex Geometry in Mathematical Physics, Presses de l'Université de Montréal, 1982.